# Semi-symmetric algebras: General Constructions. Part II

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**Abstract**

In [3] we present the construction of the semi-symmetric algebra $[\chi](E)$ of a module $E$ over a commutative ring $K$ with unit, which generalizes the tensor algebra $T(E)$, the symmetric algebra $S(E)$, and the exterior algebra $\wedge(E)$, deduce some of its functorial properties, and prove a classification theorem. In the present paper we continue the study of the semi-symmetric algebra and discuss its graded dual, the corresponding canonical bilinear form, its coalgebra structure, as well as left and right inner products. Here we present a unified treatment of these topics whose exposition in [2, A.III] is made simultaneously for the above three particular (and, without a shadow of doubt — most important) cases.

## 1 Introducton

In order to make the exposition self-contained, in this introduction we remind the main definitions and results from [3].

Let $K$ be a commutative ring with unit 1. Denote by $U(K)$ the group of units of $K$. Given a positive integer $d$, let $W \leq S_d$ be a permutation group, and let $\chi$ be a linear $K$-valued character of the group $W$, that is, a group homomorphism $\chi\colon W \to U(K)$. We call a $W$-module any $K$-linear representation of $W$ and view it also as a left unitary module over the group ring $KW$. Let $M$ be a $W$-module. We denote by ${}_{\chi}M$ the $W$-submodule of $M$, generated by all differences $\chi(\sigma)z-\sigma z$, where $\sigma \in W$, $z \in M$, and by $M_{\chi}$ the $W$-submodule of $M$, consisting of all $z \in M$ such that $\sigma z = \chi(\sigma)z$ for all $\sigma \in W$. Given $K$-modules $E$, $F$, we denote by $Mult_K(E^d, F)$ the $K$-module consisting of all $K$-multilinear maps $E \to F$, and by $T^d(E)$ — the $d$-th tensor power of $E$. The $K$-modules $T^d(E)$, $Hom_K(T^d(E), F)$, and $Mult_K(E^d, F)$ have the usual structure of $W$-modules, see [1, Ch. III, Sec. 5, $n^o$ 1]. We denote the factor- module $T^d(E)/_{\chi^{-1}}T^d(E)$ by $[\chi]^d(E)$, and call it *$d$-th semi-symmetric power of weight $\chi$ of the $K$-module $E$*. By definition, $[\chi]^0(E) = K$. The image of the tensor $x_1 \otimes \ldots \otimes x_d \in T^d(E)$ by the

canonical homomorphism $\varphi_d: T^d(E) \to [\chi]^d(E)$ is denoted by $x_1\chi \dots \chi x_d$, and is called *decomposable* $d - \chi$*-vector*. Thus, $x_{\sigma(1)}\chi \dots \chi x_{\sigma(d)} = \chi(\sigma)x_1\chi \dots \chi x_d$ for any permutation $\sigma \in W$.

In [3, (1.1.1)] we show that $d$-th semi-symmetric power $[\chi]^d(E)$ is a representing object for the functor $Mult_K(E^d, -)_\chi$. As usual, we denote by $S_\infty$ the group of all permutations of the set of all positive integers, which fix all but finitely many elements. We identify the symmetric group $S_d$ with the subgroup of $S_\infty$, consisting of all permutations fixing any $n > d$. Let $(W_d)_{d\geq 1}$ be a sequence of subgroups of $S_\infty$. This sequence is said to be *admissible* if $W_d \leq S_d$ for all $d \geq 1$. A sequence of $K$-valued characters $(\chi_d: W_d \to U(K))_{d\geq 1}$ is said to be *admissible* if its sequence of domains $(W_d)_{d\geq 1}$ is admissible. We define an injective endomorphism $\omega$ of the symmetric group $S_\infty$ by the formula $(\omega(\sigma))(d) = \sigma(d-1)+1$, $(\omega(\sigma))(1) = 1$. A sequence $(W_d)_{d\geq 1}$ is called $\omega$*-stable* if it is admissible, and $W_d \leq W_{d+1}$, $\omega(W_d) \leq W_{d+1}$, for all $d \geq 1$. A sequence of linear $K$-valued characters $(\chi_d: W_d \to U(K))_{d\geq 1}$ is said to be $\omega$*-invariant* if its sequence of domains $(W_d)_{d\geq 1}$ is $\omega$-stable, and

$$\chi_{d+1|W_d} = \chi_d = \chi_{d+1} \circ \omega_{|W_d}$$

for all $d \geq 1$. Given a $K$-module $E$, any admissible sequence of characters $\chi = (\chi_d)_{d\geq 1}$ produces a graded $K$-module $[\chi](E) = \coprod_{d\geq 0}[\chi]^d(E)$, where $[\chi]^d(E) = [\chi_d]^d(E)$, and $[\chi]^0(E) = K$. Denote by $\varphi(E)$ the canonical $K$-linear homomorphism $\coprod_{d\geq 0}\varphi_d: T(E) \to [\chi](E)$, where $\varphi_0 = id_K$. We denote by $K^{(\infty)}$ a free $K$-module with countable basis. The following two theorems are proved in [3] (see [3, 1.3.1] and [3, 1.3.3]):

**Theorem 1** *Let $\chi$ be an admissible sequence of characters. The following statements are then equivalent.*

*(i) The sequence $\chi$ is $\omega$-invariant;*

*(ii) for any $K$-module $E$ the $K$-module $[\chi](E)$ has a structure of associative graded $K$-algebra, such that $\varphi(E)$ is a homomorphism of graded $K$-algebras;*

*(iii) the $K$-module $[\chi](K^{(\infty)})$ has a structure of associative graded $K$-algebra, such that $\varphi(K^{(\infty)})$ is a homomorphism of graded $K$-algebras.*

The $K$-algebra $[\chi](E)$ is called the *semi-symmetric algebra of weight $\chi$ of the $K$-module $E$*, and its elements — $\chi$*-vectors*.

**Theorem 2** *Let $W = (W_d)_{d\geq 1}$ be an $\omega$-stable sequence of groups. Then the group of all $\omega$-invariant sequences of characters on $W$ (with componentwise multiplication) is trivial or isomorphic to the multiplicative subgroup of $K$ consisting of all involutions.*

We obtain immediately

**Corollary 3** *If $\chi = (\chi_d)_{d\geq 1}$ is an $\omega$-stable sequence of characters, then*

*(i) one has $\chi = \chi^{-1}$, where $\chi^{-1} = (\chi_d^{-1})_{d\geq 1}$;*

*(ii) if the ring $K$ is an integral domain, then the possible values of $\chi_d$ in $K$ are $\pm 1$ for any $d \geq 1$.*

When $W_d = \{1\}$ for all $d \geq 1$, the graded algebra $[\chi](E)$ coincides with the tensor algebra $T(E)$. When $W_d = S_d$ and $\chi_d$ is the unit character for all $d \geq 1$, the graded algebra $[\chi](E)$ coincides with the symmetric algebra $S(E)$. When $W_d = S_d$ and $\chi_d$ is the signature for all $d \geq 1$, the graded algebra $[\chi](E)$ is the anti-symmetric algebra of $E$; in particular, if $1/2 \in K$, then $[\chi](E)$ is the exterior algebra $\wedge(E)$ of the $K$-module $E$. If $E$ is a $n$-generated $K$-module, $k \geq n$, and if $W_d = \{1\}$ for all $d \leq k$, $W_d = S_d$ for all $d > k$, and $\chi_d$ is the signature for all $d \geq 1$, then $[\chi](E)$ is *the tensor algebra truncated by its elements of degree* $> k$.

Let $W \leq S_d$ be a permutation group and let $\chi$ be a linear $K$-valued character of the group $W$. In [5, 1] we construct a basis for the $d$-th semi-symmetric power $[\chi]^d(E)$, $d \geq 1$, starting from the standard basis for $T^d(E)$ in the case $K$ is a field of characteristics 0, but the results hold when $K$ is a commutative ring with unit, which is an integral domain, the order of the group $W$ is invertible in $K$, and the $K$-module $E$ is free, see [4] where this generalization was announced. The counterexamples from [4] show that these conditions are necessary for $[\chi]^d(E)$ to be a free $K$-module for all permutation groups $W \leq S_d$ and for all characters $\chi\colon W \to U(K)$. Here we prove these general results, see Theorem 5, its Corollary 6, and Example 10.

In this paper we continue the study of semi-symmetric algebras under the condition that the commutative ring $K$ is both a $\mathbb{Q}$-ring and an integral domain, and under the assumption that the $K$-module $E$ is a free $K$-module with a finite basis. We unite the bases for $[\chi]^d(E)$, $d \geq 0$, and get a basis for the semi-symmetric algebra $[\chi](E)$ of weight $\chi$, considered as a $K$-module. This is done in Corollary 9.

Further, we study some duality properties of the semi-symmetric powers and algebras of weight $\chi$. In Theorem 11 we define a non-singular bilinear form on the product $[\chi]^d(E) \times [\chi^{-1}]^d(E^*)$, and use it to identify the K-modules $([\chi]^d(E))^*$ and $[\chi^{-1}](E^*)$. Mimicking the case of an exterior power, we make use of generalized Schur function (see [6]) instead of determinant. After this identification, the above bilinear form coincides with the canonical bilinear form of the $K$-module $[\chi]^d(E)$; here $M^*$ denotes the dual of the $K$-module $M$. Thus, we get an identification of the semi-symmetric algebra $[\chi](E^*)$ with the dual graded algebra $([\chi](E))^{*gr}$ of the semi-symmetric algebra $[\chi](E)$, see Theorem 16, (i). Moreover, we extend the sequence of the above canonical bilinear forms to the canonical bilinear form of the graded algebras $[\chi](E)$ and $([\chi](E))^{\otimes k}$, by assuming that the homogeneous components are orthogonal, see Theorem 16, (ii), (iii). Because of the above identification, the elements of the semi-symmetric algebra $[\chi](E^*)$ are called *$\chi$-forms*. In Corollary 22 we define a structure of graded coassociative and counital $K$-coalgebra on $[\chi](E)$, and show that the structure of graded associative algebra with unit on its dual $([\chi](E))^{*gr} = [\chi](E^*)$, defined by functoriality, coincide with the usual structure of graded associative algebra with unit on the graded $K$-module $[\chi](E^*)$. In particular, when $[\chi](E)$ is the graded $K$-module underlaying the symmetric algebra (or the exterior algebra, or the tensor algebra) of the $K$-module $E$, we obtain the usual structure of $K$-coalgebra on it (see [2, A III, 139-141]). In Section 5, following [1, Ch. III, Sec.

8, $n^o$ 4], we find out the main properties of the left and right inner products of a $\chi$-vector and a $\chi$-form.

## 2 Basis of semi-symmetric algebra of a free module

Let $W$ be a a finite group, and let $\chi$ be a linear $K$-valued character of the group $W$. Let us assume that $|W| \in U(K)$ and set $a_\chi = |W|^{-1} \sum_{\sigma\in W} \chi^{-1}(\sigma)\sigma$. The element $a_\chi$ of the group ring $KW$ defines $K$-linear endomorphism $a_\chi\colon M \to M$ by the rule $z \mapsto a_\chi z$. Then the $W$-submodule ${}_\chi M$ of $M$ is the kernel of $a_\chi$, and the $W$-submodule $M_\chi$ of $M$ is the image of $a_\chi$.

Let $M$ be a free $K$-module with basis $(e_i)_{i\in I}$. Let us suppose that the finite group $W$ acts on the index set $I$. Denote by $W_i$ the stabilizer of $i \in I$ and by $W^{(i)}$ a system of representatives of the left classes of $W$ modulo $W_i$. Let $(\gamma_i)_{i\in I}$ be a family of maps $W \to U(K)$ such that $\gamma_i(\sigma\tau) = \gamma_{\tau i}(\sigma)\gamma_i(\tau)$ for all $i \in I$, and all $\sigma, \tau \in W$. In particular, the restriction of $\gamma_i$ on $W_i$ is a $K$-valued character of the group $W_i$ for any $i \in I$. The $K$-module $M$ has a structure of monomial $W$-module, defined by the rule

$$\sigma e_i = \gamma_i(\sigma)e_{\sigma i},\ \sigma \in W,\ i \in I. \tag{1}$$

We set $I(\chi, M) = \{i \in I \mid \gamma_i = \chi \text{ on } W_i\}$, $I_0(\chi, M) = I\backslash I(\chi, M)$.

**Lemma 4** *(i) The set $I(\chi, M)$ is a $W$-stable subset of $I$;*
*(ii) one has $a_\chi(v_i) = 0$ for $i \in I_0(\chi, M)$.*

**Proof:** (i) Given $i \in I$, suppose $\sigma \in W$ and $\tau \in W_i$. Then $W_{\sigma i} = \sigma W_i \sigma^{-1}$ and $\chi(\sigma\tau\sigma^{-1}) = \chi(\tau)$. Moreover,

$$\gamma_{\sigma i}(\sigma\tau\sigma^{-1}) = \gamma_{\sigma^{-1}\sigma i}(\sigma\tau)\gamma_{\sigma i}(\sigma^{-1}) = \gamma_{\sigma i}(\sigma^{-1})\gamma_i(\sigma\tau) =$$

$$\gamma_{\sigma\tau i}(\sigma^{-1})\gamma_i(\sigma\tau) = \gamma_i(\sigma^{-1}\sigma\tau) = \gamma_i(\tau).$$

(ii) The complement of $I(\chi, M)$ in $I$ also is $W$-stable; let $i \in I\backslash I(\chi, M)$. We have

$$a_\chi(v_i) = |W|^{-1} \sum_{\sigma\in W^{(i)}} \sum_{\tau\in W_i} \chi^{-1}(\sigma\tau)\gamma_i(\sigma\tau)v_{\sigma\tau i} =$$

$$|W|^{-1} \sum_{\sigma\in W^{(i)}} \chi^{-1}(\sigma)\gamma_i(\sigma)(\sum_{\tau\in W_i} \chi^{-1}(\tau)\gamma_i(\tau))v_{\sigma i},$$

and the equality $a_\chi(v_i) = 0$ holds because the product $\chi^{-1}\gamma_i$ is not the unit character of the group $W_i$.

We choose an element $i$ from any $W$-orbit in $I$ and denote the set of these $i$'s by $I^*$. Finally, we set $J(\chi, M) = I^* \cap I(\chi, M)$, and $J_0(\chi, M) = I^* \cap I_0(\chi, M)$.

Following [2, Ch. III, Sec. 5, $n^o$ 4], we get a basis of the $K$-module $M$ consisting of

$$e_j, \quad j \in J(\chi, M), \tag{2}$$
$$e_i - \chi(\sigma)\gamma_i(\sigma)e_{\sigma i}, \quad i \in I^*, \ \sigma \in W^{(i)}, \ \sigma \notin W_i, \tag{3}$$
$$e_i, \quad i \in J_0(\chi, M). \tag{4}$$

**Theorem 5** *Let the ring $K$ be an integral domain and let $|W| \in U(K)$. Then*
*(i) the union of the families (3) and (4) is a basis for ${}_\chi M$;*
*(ii) the family $a_\chi(e_j)$, $j \in J(\chi, M)$, is a basis for $M_\chi$;*
*(iii) the family $e_j$ mod$({}_\chi M)$, $j \in J(\chi, M)$, is a a basis for the factor-module $M/{}_\chi M$.*

**Proof:** (i) The family (3) is in ${}_\chi M$ by definition. Lemma 4, (ii), implies that the family (4) is contained in ${}_\chi M$. Now, set $J = J(\chi, M)$ and suppose that $\sum_{j\in J} k_j a_\chi(v_j) = 0$ for some $k_j \in K$ such that $k_j = 0$ for all but a finite number of indices $j \in J$. We have

$$\sum_{j\in J} k_j a_\chi(v_j) = |W|^{-1} \sum_{j\in J} \sum_{\sigma \in W^{(j)}} k_j |W_j| \chi^{-1}(\sigma)\gamma_j(\sigma) v_{\sigma j},$$

hence $k_j = 0$ for all $j \in J$, which proves part (i). In addition, we have proved that the elements $a_\chi(v_j)$, $j \in J(\chi, M)$, are linearly independent.

(ii) The elements $a_\chi(v_j)$, $j \in J(\chi, M)$, are in $M_\chi$ and, moreover, each element of $M_\chi$ has the form $a_\chi(z)$ for some $z \in M$. Since the union of families (2) – (4) is a basis for $M$ and since the endomorphism $a_\chi$ annihilates (3) and (4), part (ii) holds.

(iii) Part (ii) implies part (iii).

Now, let us suppose that the $K$-module $E$ has basis $(e_\ell)_{\ell\in L}$. Then the tensor power $M = T^d(E)$ has basis $(e_i)_{i\in L^d}$, and if $W \leq S_d$ is a permutation group, the rule $\sigma e_i = e_{\sigma i}$, $\sigma \in W$, defines on $M$ a structure of monomial $W$-module.

**Corollary 6** *Let $W \leq S_d$ be a permutation group and let $\chi$ be a linear $K$-valued character of $W$. If $K$ is an integral domain and $|W| \in U(K)$, then the $d$-th semi-symmetric power $[\chi](E)$ of weight $\chi$ of a free $K$-module $E$ with basis $(e_\ell)_{\ell\in L}$ is a free $K$-module with basis*

$$(e_{j_1}\chi \dots \chi e_{j_d})_{(j_1,\dots,j_d)\in J(\chi,T^d(E))}.$$

**Proof:** Substitute $M = T^d(E)$, $I = L^d$, $\gamma_i(\sigma) = 1$ for all $\sigma \in W$, $i \in L^d$, in Theorem 5.

**Corollary 7** *Let $W \leq S_d$ be a permutation group and let $\chi$ be a linear $K$-valued character of $W$. If $K$ is an integral domain, $|W| \in U(K)$, and if $E$ is a projective $K$-module (a projective $K$-module of finite type), then the $d$-th semi-symmetric power $[\chi](E)$ of weight $\chi$ is a projective $K$-module (a projective $K$-module of finite type).*

**Proof:** Let $L$ be a set (a finite set), and let $K^{(L)}$ be the free $K$-module with the canonical basis indexed by $L$. Let

$$0 \to E \to K^{(L)}$$

be a splitting monomorphism of $K$-modules. Since the functor $[\chi]^d(-)$ transforms epimorphisms into epimorphisms, the sequence

$$0 \to [\chi]^d(E) \to [\chi]^d(K^{(L)})$$

also is a splitting monomorphism of $K$-modules, and, moreover, according to Corollary 6, $[\chi]^d(K^{(L)})$ is a free $K$-module (free module with finite basis). Therefore $[\chi]^d(E)$ is a projective $K$-module (a projective $K$-module of finite type).

**Remark 8** *Let us set $J\left(\chi, T^0(E)\right) = \{\emptyset\}$, $e_\emptyset = 1$. We unite the bases of all semi-symmetric powers $[\chi]^d(E)$ (see Corollary 6), thus getting $J(\chi, T(E)) = \cup_{d\geq 0} J\left(\chi_d, T^d(E)\right)$. In particular, when $L = [1, n]$, the elements of the set $J\left(\chi, T^d(E)\right)$ can be chosen to be lexicographically minimal in their $W$-orbits, and we can introduce following notation:*

$$I\left(\chi, T^d(E)\right) = I(\chi, n, d), \; I_0\left(\chi, T^d(E)\right) = I_0(\chi, n, d),$$

$$J\left(T^d(E), \chi\right) = J(\chi, n, d), \; J(T(E), \chi) = J(\chi, n).$$

*For any $i \in I(\chi, n, d)$ we define $\ell m(i)$ to be the lexicographically minimal element in the $W$-orbit of $i$, and set $\zeta(i) = \chi_d(\sigma)$, where $\sigma \in W_d$ is such that $\sigma i = \ell m(i)$. Since the restriction of the character $\chi_d$ is identically $1$ on the stabilizer $(W_d)_i$, the element $\zeta(i) \in U(K)$ does not depend on the choice of $\sigma$.*

Let $\chi = (\chi_d)_{d\geq 1}$ be an $\omega$-invariant sequence of characters and let $W = (W_d)_{d\geq 1}$ be the sequence of their domains.

**Corollary 9** *Let $K$ be both a $\mathbb{Q}$-ring and an integral domain.*

*(i) If $E$ is a $K$-module with basis $(e_\ell)_{\ell\in L}$, then the family $(e_j)_{j\in J(T(E),\chi)}$ is a basis for the semi-symmetric algebra $[\chi](E)$ of weight $\chi$, considered as a $K$-module;*

*(ii) If $E$ is a $K$-module with finite basis $(e_\ell)_{\ell=1}^n$, then the family $(e_j)_{j\in J(\chi,n)}$ is a basis for the semi-symmetric algebra $[\chi](E)$ of weight $\chi$, considered as a $K$-module. If $j \in J(\chi, n, d)$ and $k \in J(\chi, n, e)$, then the multiplication table of the $K$-algebra $[\chi](E)$ is given by the formulae*

$$e_j \chi e_k = \begin{cases} 0 & \text{if } (j,k) \in I_0(\chi, n, d+e) \\ \zeta(j,k) e_{\ell m(j,k)} & \text{if } (j,k) \in I(\chi, n, d+e). \end{cases}$$

**Proof:** (i) Straightforward use of Corollary 6.

(ii) The first part is a particular case of (i). We have $e_j \chi e_k = e_{(j,k)}$, and in case $(j,k) \in I_0(\chi, n, d+e)$ Lemma 4, (ii), implies $e_{(j,k)} = 0$. Otherwise, $e_{\ell m(j,k)} \in J(\chi, n, d+e)$, and we make use of Remark 8.

**Example 10** *We will show that if some of the conditions of Corollary 6 fail, then the $K$-module $[\chi]^d(E)$ is not necessarily free.*

*(i) The ring $K$ is not an integral domain.*

*We set $K = \mathbb{Z}_{15}$, $W = \{(1), (12)(34), (13)(24), (14)(23)\} \leq S_4$ is the Klein four group, $\chi((12)(34)) = 4$, $\chi((13)(24)) = 4$, $\chi((14)(23)) = 1$, $E = Ke_1 \coprod Ke_2$, $I = [1,2]^4$, $e_i = e_{i_1} \otimes \ldots \otimes e_{i_4}$ for $i = (i_1, \ldots, i_4) \in I$. We have $\chi = \chi^{-1}$. The $K$-module $T^4(E)_\chi$ is spanned by the elements*

$$a_\chi(e_{(1,1,1,1)}) = 10e_{(1,1,1,1)},$$
$$a_\chi(e_{(2,2,2,2)}) = 10e_{(2,2,2,2)},$$
$$a_\chi(e_{(1,1,2,2)}) = 5e_{(1,1,2,2)} + 5e_{(2,2,1,1)},$$
$$a_\chi(e_{(1,2,1,2)}) = 5e_{(1,2,1,2)} + 5e_{(2,1,2,1)},$$
$$a_\chi(e_{(1,2,2,1)}) = 8e_{(1,2,2,1)} + 8e_{(2,1,1,2)},$$
$$a_\chi(e_{(1,1,1,2)}) = e_{(1,1,1,2)} + e_{(1,1,2,1)} + e_{(1,2,1,1)} + 4e_{(2,1,1,1)},$$
$$a_\chi(e_{(1,2,2,2)}) = e_{(1,2,2,2)} + e_{(2,1,2,2)} + e_{(2,2,1,2)} + 4e_{(2,2,2,1)}.$$

*Thus, the $\mathbb{Z}_{15}$-module $M_\chi$ is isomorphic to the submodule*

$$\mathbb{Z}_{15}e^{(1)} \coprod \mathbb{Z}_{15}e^{(2)} \coprod \mathbb{Z}_{15}e^{(3)} \coprod \mathbb{Z}_{15}10e^{(4)} \coprod \mathbb{Z}_{15}10e^{(5)} \coprod \mathbb{Z}_{15}5e^{(6)} \coprod \mathbb{Z}_{15}5e^{(7)}$$

*of a free $\mathbb{Z}_{15}$-module with 7 generators $e^{(1)}, \ldots, e^{(7)}$. This submodule has $15^3 3^4$ elements, and this number is not a power of 15, hence $[\chi]^4(E) = M/_\chi M$ is not a free $\mathbb{Z}_{15}$-module.*

*(ii) The order $|W|$ of the group $W$ is not invertible in the ring $K$.*

*We denote by $\varepsilon$ a primitive 3-th root of unity and set $K = \mathbb{Z}[\varepsilon]$, $W = \{(1), (123), (132)\} \leq S_3$, $\chi(123) = \varepsilon$, $E = Ke_1 \coprod Ke_2$, $M = T^3(E)$, $I = [1,2]^3$, $e_i = e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$ for $i = (i_1, i_2, i_3) \in I$. The $K$-module $[\chi^2]^3(E) = M/_\chi M$ is spanned by the elements*

$$e_{(1,1,1)}, e_{(2,2,2)}, e_{(1,1,2)}, e_{(1,2,2)} \pmod{{}_\chi M}.$$

*Suppose that for some $k_1, \ldots, k_4 \in K$ we have*

$$k_1 e_{(1,1,1)} + k_2 e_{(2,2,2)} + k_3 e_{(1,1,2)} + k_4 e_{(1,2,2)} \in {}_\chi M. \tag{5}$$

*Applying the operator of $\chi$-symmetry $A_\chi = \sum_{\sigma \in W} \chi^2(\sigma)\sigma$, we obtain*

$$k_1 A_\chi e_{(1,1,1)} + k_2 A_\chi e_{(2,2,2)} + k_3 A_\chi e_{(1,1,2)} + k_4 A_\chi e_{(1,2,2)} = 0.$$

*On the other hand, $A_\chi e_{(1,1,1)} = A_\chi e_{(2,2,2)} = 0$, and $A_\chi e_{(1,1,2)}$ and $A_\chi e_{(1,2,2)}$ are linearly independent over $K$, hence $k_3 = k_4 = 0$. Thus,*

$$k_1 e_{(1,1,1)} + k_2 e_{(2,2,2)} = \ell_1(1-\varepsilon)e_{(1,1,1)} + \ell_2(1-\varepsilon)e_{(2,2,2)} + f,$$

*where $\ell_1, \ell_2 \in K$, and $f$ is a $K$-linear combination of the tensors $e_{(1,1,2)} - \varepsilon e_{(2,1,1)}$, $e_{(1,1,2)} - \varepsilon^2 e_{(1,2,1)}$, $e_{(1,2,2)} - \varepsilon e_{(2,1,2)}$, and $e_{(1,2,2)} - \varepsilon^2 e_{(2,2,1)}$, that is, $k_1 \in (1-\varepsilon)K$, $k_2 \in (1-\varepsilon)K$, and $f = 0$. Therefore, (5) is equivalent to $k_1 \in (1-\varepsilon)K$, $k_2 \in (1-\varepsilon)K$, and $k_3 = k_4 = 0$. In particular, the $K$-module $[\chi^2]^3(E)$ has non-zero torsion part, hence it is not free.*

## 3 Duality

Let the ring $K$ be an integral domain. Let us denote by $\mathcal{F}$ the category of $K$-modules with finite bases and, as usual, denote by $Ob(\mathcal{F})$ its set of objects. Let $E$ be a $K$-module with finite basis $(e_\ell)_{\ell=1}^n$ and let $E^*$ be the dual $K$-module with dual basis $(e_\ell^*)_{\ell=1}^n$. Denote by $\langle\ ,\ \rangle$ the canonical bilinear form $E \times E^* \to K$, $(x, x^*) \mapsto x^*(x)$. Let $W \leq S_d$ be a permutation group with $|W| \in U(K)$, and let $\chi$ be a linear $K$-valued character of $W$. We set $|W_\emptyset| = 1$. For any $d \times d$-matrix $A = (a_{ij})$ over $K$, the expression

$$d_\chi^W(A) = \sum_{\sigma \in W} \chi(\sigma) a_{\sigma^{-1}(1)1} \dots a_{\sigma^{-1}(d)d}$$

is known as (generalized) Schur function. It was introduced by I. Schur in [6].

**Theorem 11** *(i) The formulae*

$$[\chi]^d(E) \times [\chi^{-1}]^d(E^*) \to K, \tag{6}$$

$$B(x_1\chi \dots \chi x_d, x_1^*\chi^{-1} \dots \chi^{-1} x_d^*) = d_\chi^W((< x_i, x_j^* >)_{i,j=1}^d),$$

*for $d \geq 1$, and the formula*

$$[\chi]^0(E) \times [\chi^{-1}]^0(E^*) \to K, \tag{7}$$

$$B(k, k^*) = kk^*,$$

*define non-singular bilinear forms;*

*(ii) if $\iota_E^{(d)}: [\chi^{-1}]^d(E^*) \to ([\chi]^d(E))^*$ (resp., $\iota^{(0)}: [\chi^{-1}]^0(E^*) \to ([\chi]^0(E))^*$) is the isomorphism of $K$-modules, associated with (6) (resp., with (7)), then the family $\iota^{(d)} = (\iota_E^{(d)})_{E \in Ob(\mathcal{F})}$ (resp., $\iota^{(0)}$) is an isomorphism of functors, $\iota^{(d)}: [\chi^{-1}]^d(-^*) \to ([\chi]^d(-))^*$;*

*(iii) after the identifications via the functor $\iota^{(d)}$ from (ii), $B$ is the canonical bilinear form of the $K$-module $[\chi]^d(E)$, and the bases $(e_j)_{j\in J}$ and $((1/|W_j|)e_j^*)_{j \in J}$ are dual.*

**Proof:** (i) For $d = 0$ we get the multiplication of the ring $K$. Let us suppose $d \geq 1$. The product $E^d \times (E^*)^d$ has a natural structure of $W \times W$-module (see [3, 2.1]), and the map

$$E^d \times (E^*)^d \to K,$$

$$(x_1, \dots, x_d, x_1^*, \dots, x_d^*) \mapsto d_\chi^W((< x_i, x_j^* >)_{i,j=1}^d),$$

is semi-symmetric of weight $\chi$ with respect to variables $x_1, \dots, x_d$, and semi-symmetric of weight $\chi^{-1}$ with respect to variables $x_1^*, \dots, x_d^*$. Hence by [3, Lemma 2.1.2] it gives rise to a bilinear form $B$ given by formulae (6). We have $J(\chi, n, p) = J(\chi^{-1}, n, p) = J$, and in accord with Corollary 6, $(e_j)_{j\in J}$ is a basis for $[\chi]^d(E)$, and $(e_j^*)_{j\in J}$ is a basis for $[\chi^{-1}]^d(E^*)$. If $\delta(j, k)$ is Kronecker's delta, then

$$B(e_j, e_k^*) = \sum_{\sigma \in W} \chi^{-1}(\sigma) \langle e_{j_{\sigma^{-1}(1)}}, e_{k_1}^* \rangle \dots \langle e_{j_{\sigma^{-1}(d)}}, e_{k_d}^* \rangle$$

$$= \sum_{\sigma \in W} \chi^{-1}(\sigma)\delta(j_{\sigma^{-1}(1)}, k_1)\dots\delta(j_{\sigma^{-1}(d)}, k_d),$$

hence

$$B(e_j, e_k^*) = |W_j|\delta(j,k). \tag{8}$$

In particular, (6) and (7) are non-singular forms for any $d \geq 0$.

(ii) For any $K$-linear map $u: E \to F$ we denote by ${}^t u: F^* \to E^*$ its transpose. A direct computation shows that

$${}^t([\chi]^d(u)) \circ \iota_F^{(d)} = \iota_E^{(d)} \circ ([\chi^{-1}]^d({}^t u)). \tag{9}$$

(iii) The equality (8) yields that $(e_j)_{j\in J}$, $(\frac{1}{|W_j|}e_j^*)_{j\in J}$ is a pair of dual bases.

**Remark 12** *Throughout the end of the paper we will use notation*

$$\langle x_1\chi\dots\chi x_d, x_1^*\chi^{-1}\dots\chi^{-1}x_d^*\rangle = B(x_1\chi\dots\chi x_d, x_1^*\chi^{-1}\dots\chi^{-1}x_d^*),$$

*and in this notation, for any $x = \sum_{j\in J} x_j e_j$, and for any $x^* = \sum_{j\in J} x_j^* e_j^*$, one has*

$$\langle x, x^*\rangle = \sum_{j\in J} |W_j| x_j x_j^*. \tag{10}$$

**Remark 13** *In accord with Theorem 11, (ii), (iii), for any $d \geq 1$, and for any $K$-module $E$ with finite basis we identify $([\chi]^d(E))^*$ with $[\chi^{-1}]^d(E^*)$ as $K$-modules via the functor $\iota^{(d)}$, and call the elements of $[\chi^{-1}]^d(E^*)$ $d-\chi$-forms on $E$.*

**Corollary 14** *For any $K$-linear map $u: E \to F$ one has ${}^t([\chi]^d u) = [\chi^{-1}]^d({}^t u)$.*

**Proof:** This the equality (9) after the identifications via the functor $\iota^{(d)}$.

Let $A = (a_{r,s})$ be an $m \times n$ matrix over $K$ and let $d \geq 1$. For any $j \in J(\chi, m, d)$, $k \in J(\chi, n, d)$, we set $a_{jk} = \prod_{t=1}^d a_{j_t k_t}$, and

$$A_{(j)k}(\chi) = \sum_{\tau \in W^{(j)}} \chi^{-1}(\tau) a_{\tau j k},$$

and call the expression $A_{(j)k}(\chi)$ the $(j,k)$*-th row minor of weight* $\chi$ of $A$.

Let $A = (a_{rt})$ and $A' = (a'_{sh})$ be two $n \times d$ matrices over $K$. Using notation from the beginning of Section 3, we set $x_t = \sum_{r=1}^n a_{rt}e_r$, $x_h^* = \sum_{s=1}^n a'_{sh}e_s^*$, where $t, h = 1, \dots, d$. Then $\langle x_t, x_h^*\rangle = \sum_{r=1}^n a_{rt}a'_{rh}$ is the $th$-entry of the matrix ${}^tAA'$, and hence

$$\langle x_1\chi\dots\chi x_d, x_1^*\chi^{-1}\dots\chi^{-1}x_d^*\rangle = d_\chi({}^tAA'). \tag{11}$$

On the other hand,

$$x_1\chi\dots\chi x_d = \sum_{j\in J} A_{(j)}(\chi)e_j,\ x_1^*\chi^{-1}\dots\chi^{-1}x_d^* = \sum_{j\in J} A'_{(j)}(\chi^{-1})e_j^*, \tag{12}$$

where $A_{(j)}(\chi) = A_{(j)k}(\chi)$, and $A'_{(j)}(\chi^{-1}) = A'_{(j)k}(\chi^{-1})$ with $k = (1, ..., d)$. Therefore (10) and (11) yield

$$d_\chi({}^tAA') = \sum_{j\in J} |W_j| A_{(j)}(\chi) A'_{(j)}(\chi^{-1}).$$

In particular, when $A = A'$ we obtain generalized Lagrange identity

$$d_\chi({}^tAA) = \sum_{j\in J} |W_j| A_{(j)}(\chi) A_{(j)}(\chi^{-1}).$$

**Lemma 15** *Let $A = (a_{th})$ be a $d \times d$ matrix over $K$. Then, in the previous notations, one has:*

*(i) $d_\chi({}^tA) = d_{\chi^{-1}}(A)$;*

*(ii) $d_\chi(A) = \langle x_1\chi^{-1} \dots \chi^{-1}x_d, e_1^*\chi \dots \chi e_d^*\rangle$;*

*(iii) The generalized Schur function $d_\chi(A)$ is semi-symmetric of weight $\chi^{-1}$ (resp., of weight $\chi$) with respect to the columns (resp., the rows) of the matrix $A$.*

**Proof:** (i) Direct checking.

(ii) Using (i) and (11) with $d = n$ and $A' = I_d$ (the unit $d \times d$ matrix), we obtain the equality.

(iii) This is an immediate consequence of (ii) and (i).

Throughout the end of the paper we fix the following notation:

$K$ is both a $\mathbb{Q}$-ring and an integral domain;

$(\chi_d\colon W_d \to K)_{d\geq 1}$ is an $\omega$-invariant sequence of characters;

$E$ is a $K$-module with finite basis;

$[\chi](E)$ is the semi-symmetric algebra of weight $\chi$ of $E$.

We remind that the dual graded $K$-module $([\chi](E))^{*gr}$ is, by definition, the direct sum $\coprod_{d\geq 0}([\chi]^d(E))^*$, where we identify a linear form on $[\chi]^d(E)$ with its extension by 0 to $[\chi](E)$. Let us set $\iota = \coprod_{d\geq 0} \iota^{(d)}$.

Since the $K$-module $E$ has a finite basis, then it is a projective module of finite type, and using Corollary 7, [2, A II, p. 80, Cor. 1], and Theorem 11, we obtain

**Theorem 16** *(i) $\iota\colon [\chi](-^*) \to ([\chi](-))^{*gr}$ is an isomorphism of functors;*

*(ii) After the identification via the functor $\iota$ from (i), the restriction of the canonical bilinear form of the $K$-module $[\chi](E)$ on $[\chi](E) \times [\chi](E^*)$ is given by the formulae*

$$\langle\ ,\ \rangle\colon [\chi](E) \times [\chi](E^*) \to K, \tag{13}$$

$$\langle x_1\chi \dots \chi x_r, x_1^*\chi \dots \chi x_s^*\rangle = \begin{cases} 0 & \text{if } r \neq s \\ d_\chi((\langle x_i, x_j^*\rangle)_{i,j=1}^r) & \text{if } r = s \geq 1 \\ 1 & \text{if } r = s = 0; \end{cases}$$

*(iii) for any $k \geq 2$ the restriction of the canonical bilinear form of the $K$-module $([\chi](E))^{\otimes k}$ on $([\chi](E))^{\otimes k} \times [\chi](E^*))^{\otimes k}$ is given by the formulae*

$$\langle\ ,\ \rangle\colon ([\chi](E))^{\otimes k} \times [\chi](E^*))^{\otimes k} \to K, \tag{14}$$

$$\langle x_1\chi\ldots\chi x_r\otimes x_1\chi\ldots\chi x_{r'}\otimes\cdots, x_1^*\chi\ldots\chi x_s^*\otimes x_1^*\chi\ldots\chi x_{s'}^*\otimes\cdots\rangle =$$
$$\begin{cases} 0 & \textit{if } (r,r',\ldots)\neq(s,s',\ldots)\\ \langle x_1\chi\ldots\chi x_r, x_1^*\chi\ldots\chi x_r^*\rangle\langle x_1\chi\ldots\chi x_{r'}, x_1^*\chi\ldots\chi x_{r'}^*\rangle\ldots & \textit{if } (r,r',\ldots)=(s,s',\ldots),\end{cases}$$

**Remark 17** *Let $d$, $e\ldots$, $h$, be non-negative integers with $d+e+\cdots+h=n$. We set*

$$J(\chi;n;d,e,\ldots,h)=$$
$$\{(j,k,\ldots,r)\in J(\chi,n,d)\times J(\chi,n,e)\times\cdots\times J(\chi,n,h)\mid \ell m(j,k,\ldots,r)=(1,...,n)\}.$$

*Let $M(\chi;n;d,e,\ldots,h)$ be the set of lexicographically minimal representatives of left classes of $W_n$ modulo $W_d\times\omega^d(W_e)\times\cdots\times\omega^{d+e+\cdots}(W_h)$. We identify the set $M(\chi;n;d,e,\ldots,h)$ with the set $J(\chi;n;d,e,\ldots,h)$ via the canonical bijection*

$$M(\chi;n;d,e,\ldots,h)\to J(\chi;n;d,e,\ldots,h),$$
$$\zeta\mapsto((\zeta(1),\ldots,\zeta(d)),(\zeta(d+1),\ldots,\zeta(d+e)),\ldots,(\zeta(d+e+\cdots+1),\ldots,\zeta(n))).$$

We fix $(\lambda,\mu,\ldots,\nu)\in J(\chi;n;d,e,\ldots,h)$, and let $\sigma\in W_n$ be a permutation, such that $\lambda=(\sigma(1),\ldots,\sigma(d))$, $\mu=(\sigma(d+1),\ldots,\sigma(d+e))$, $\ldots$, and $\nu=(\sigma(d+e+\cdots+1),\ldots,\sigma(n))$. We have $\zeta(\lambda,\mu,\ldots,\nu)=\chi(\sigma)$. Let us write $d_\chi(A)$ for $d_{\chi_n}(A)$.

**Proposition 18** *Let $A$ be an $n\times n$ matrix over $K$. Then*

$$d_\chi(A)=$$
$$\zeta(\lambda,\mu,\ldots,\nu)\sum_{(j,k,\ldots,r)\in J(\chi;n;d,e,\ldots,h)}\zeta(j,k,\ldots,r)A_{(j)\lambda}(\chi)A_{(k)\mu}(\chi)\ldots A_{(r)\nu}(\chi).$$

*(Laplace expansion of $d_\chi(A)$ with respect to $\lambda$, $\mu$, $\ldots$, $\nu$).*

**Proof:** Indeed, using Lemma 15, (ii), Corollary 3, (i), the expansions (12), and Corollary 9, (ii), we obtain

$$d_\chi(A)=\langle x_1\chi^{-1}\ldots\chi^{-1}x_n, e_1^*\chi\ldots\chi e_n^*\rangle=\langle x_1\chi\ldots\chi x_n, e_1^*\chi\ldots\chi e_n^*\rangle=$$
$$\zeta(\lambda,\mu,\ldots,\nu)\langle x_{\lambda_1}\chi\ldots\chi x_{\lambda_d}\chi x_{\mu_1}\chi\ldots\chi x_{\mu_e}\chi x_{\nu_1}\chi\ldots\chi x_{\nu_h}, e_1^*\chi\ldots\chi e_n^*\rangle=$$
$$\zeta(\lambda,\mu,\ldots,\nu)\langle\sum_{(j,k,\ldots,r)\in J(\chi,n,d)\times J(\chi,n,e)\times\cdots\times J(\chi,;n,h)}\zeta(j,k,\ldots,r)$$
$$A_{(j)\lambda}(\chi)A_{(k)\mu}(\chi)\ldots A_{(r)\nu}(\chi)e_{\ell m(j,k,\ldots,r)}, e_1^*\chi\ldots\chi e_n^*\rangle=$$
$$\zeta(\lambda,\mu,\ldots\nu)\sum_{(j,k,\ldots,r)\in J(\chi;n;d,e,\ldots,h)}\zeta(j,k,\ldots,r)A_{(j)\lambda}(\chi)A_{(k)\mu}(\chi)\ldots A_{(r)\nu}(\chi).$$

**Proposition 19** *For any non-negative integers d, e,..., h with $d+e+\cdots+h=n$ one has the following expansions of the bilinear form (13) (Laplace expansions):*

$$\langle x_1\chi\dots\chi x_n, x_1^*\chi\dots\chi x_n^*\rangle =$$

$$\sum_{\zeta\in M(\chi;n;d,e,\dots,h)} \chi(\zeta)\langle x_{\zeta(1)}\chi\dots\chi x_{\zeta(d)}, x_1^*\chi\dots\chi x_d^*\rangle$$

$$\langle x_{\zeta(d+1)}\chi\dots\chi x_{\zeta(d+e)}, x_{d+1}^*\chi\dots\chi x_{d+e}^*\rangle$$

$$\cdots\langle x_{\zeta(d+e+\cdots+1)}\chi\dots\chi x_{\zeta(n)}, x_{d+e+\cdots+1}^*\chi\dots\chi x_n^*\rangle =$$

$$\sum_{\zeta\in M(\chi;n;d,e,\dots,h)} \chi(\zeta)\langle x_1\chi\dots\chi x_d, x_{\zeta(1)}^*\chi\dots\chi x_{\zeta(d)}^*\rangle$$

$$\langle x_{d+1}\chi\dots\chi x_{d+e}, x_{\zeta(d+1)}^*\chi\dots\chi x_{\zeta(d+e)}^*\rangle$$

$$\cdots\langle x_{d+e+\cdots+1}\chi\dots\chi x_n, x_{\zeta(d+e+\cdots+1)}^*\chi\dots\chi x_{\zeta(n)}^*\rangle.$$

**Proof:** We have

$$\langle x_1\chi\dots\chi x_n, x_1^*\chi\dots\chi x_n^*\rangle =$$

$$\sum_{\zeta'\in W_n}\chi(\zeta')(\langle x_{\zeta'(1)}, x_1^*\rangle\cdots\langle x_{\zeta'(d)}, x_d^*\rangle)(\langle x_{\zeta'(d+1)}, x_{d+1}^*\rangle\cdots\langle x_{\zeta'(d+e)}, x_{d+e}^*\rangle)$$

$$\cdots(\langle x_{\zeta'(d+e+\cdots+1)}, x_{d+e+\cdots+1}^*\rangle\cdots\langle x_{\zeta'(n)}, x_n^*\rangle) =$$

$$\sum_{\zeta\in M(\chi;n;d,e,\dots,h)}\ \sum_{(\sigma',\tau',\dots,\eta')\in W_d\times\omega^d(W_e)\times\cdots\times\omega^{d+e+\cdots}(W_h)} \chi(\zeta)\chi(\sigma')\chi(\tau')\dots\chi(\eta')$$

$$(\langle x_{\zeta(\sigma'(1))}, x_1^*\rangle\cdots\langle x_{\zeta(\sigma'(d))}, x_d^*\rangle)(\langle x_{\zeta(\tau'(d+1))}, x_{d+1}^*\rangle\cdots\langle x_{\zeta(\tau'(d+e))}, x_{d+e}^*\rangle)$$

$$\cdots(\langle x_{\zeta(\eta'(d+e+\cdots+1))}, x_{d+e+\cdots+1}^*\rangle\cdots\langle x_{\zeta(\eta'(n))}, x_n^*\rangle) =$$

$$\sum_{\zeta\in M(\chi;n;d,e,\dots,h)}\chi(\zeta)(\sum_{\sigma'\in W_d}\chi(\sigma')\langle x_{\zeta(\sigma'(1))}, x_1^*\rangle\cdots\langle x_{\zeta(\sigma'(d))}, x_d^*\rangle)$$

$$(\sum_{\tau'\in\omega^d(W_e)}\chi(\tau')\langle x_{\zeta(\tau'(d+1))}, x_{d+1}^*\rangle\cdots\langle x_{\zeta(\tau'(d+e))}, x_{d+e}^*\rangle)$$

$$\cdots(\sum_{\eta'\in\omega^{d+e+\cdots}(W_d)}\chi(\eta')\langle x_{\zeta(\eta'(d+e+\cdots+1))}, x_{d+e+\cdots+1}^*\rangle\cdots\langle x_{\zeta(\eta'(n))}, x_n^*\rangle) =$$

$$\sum_{\zeta\in M(\chi;n;d,e,\dots,h)} \chi(\zeta)\langle x_{\zeta(1)}\chi\dots\chi x_{\zeta(d)}, x_1^*\chi\dots\chi x_d^*\rangle$$

$$\langle x_{\zeta(d+1)}\chi\dots\chi x_{\zeta(d+e)}, x_{d+1}^*\chi\dots\chi x_{d+e}^*\rangle\cdots$$

$$\langle x_{\zeta(d+e+\cdots+1)}\chi\dots\chi x_{\zeta(n)}, x_{d+e+\cdots+1}^*\chi\dots\chi x_n^*\rangle.$$

For the second equality, we can write

$$\langle x_1\chi\dots\chi x_n, x_1^*\chi\dots\chi x_n^*\rangle =$$

$$\sum_{\zeta'\in W_n}\chi(\zeta')(\langle x_1, x^*_{\zeta'^{-1}(1)}\rangle\cdots\langle x_d, x^*_{\zeta'^{-1}(d)}\rangle)(\langle x_{d+1}, x^*_{\zeta'^{-1}(d+1)}\rangle\cdots\langle x_{d+e}, x^*_{\zeta'^{-1}(d+e)}\rangle)$$

$$\cdots(\langle x_{d+e+\cdots+1}, x^*_{\zeta'^{-1}(d+e+\cdots+1)}\rangle\cdots\langle x_n, x^*_{\zeta'^{-1}(n)}\rangle) =$$

$$\sum_{\zeta'\in W_n}\chi(\zeta')(\langle x_1, x^*_{\zeta'(1)}\rangle\cdots\langle x_d, x^*_{\zeta'(d)}\rangle)(\langle x_{d+1}, x^*_{\zeta'(d+1)}\rangle\cdots\langle x_{d+e}, x^*_{\zeta'(d+e)}\rangle)$$

$$\cdots(\langle x_{d+e+\cdots+1}, x^*_{\zeta'(d+e+\cdots+1)}\rangle\cdots\langle x_n, x^*_{\zeta'(n)}\rangle),$$

and then we proceed by analogy.

According to Lemma 31, for any $n$ we obtain a $K$-linear map

$$[\chi]^n(E)\to\oplus_{d+e+\cdots+h=n}[\chi]^d(E)\otimes[\chi]^e(E)\ldots\otimes[\chi]^h(E),$$

$$x_1\chi\ldots\chi x_n\mapsto\sum_{d+e+\cdots+h=n}\ \sum_{\rho\in M(\chi;n;d,e,\ldots,h)}\chi(\rho)$$

$$(x_{\rho(1)}\chi\ldots\chi x_{\rho(d)})\otimes(x_{\rho(d+1)}\chi\ldots\chi x_{\rho(d+e)})\otimes\ldots\otimes(x_{\rho(d+e+\cdots+1)}\chi\ldots\chi x_{\rho(n)}).$$

Therefore, for any $k\geq 2$ we get a homomorphism of graded $K$-modules

$$c_k(E)\colon[\chi](E)\to([\chi](E))^{\otimes k}, \tag{15}$$

$$c_k(E)(x_1\chi\ldots\chi x_n)=\sum_{d+e+\cdots+h=n}\ \sum_{\rho\in M(\chi;n;d,e,\ldots,h)}\chi(\rho)$$

$$(x_{\rho(1)}\chi\ldots\chi x_{\rho(d)})\otimes(x_{\rho(d+1)}\chi\ldots\chi x_{\rho(d+e)})\otimes\ldots\otimes(x_{\rho(d+e+\cdots+1)}\chi\ldots\chi x_{\rho(n)}).$$

**Corollary 20** *For any $k$ in number non-negative integers $d$, $e$,…, $h$ with $d+e+\cdots+h=n$ one has*

$$\langle x_1\chi\ldots\chi x_n, x_1^*\chi\ldots\chi x_n^*\rangle=$$

$$\langle c_k(E)(x_1\chi\ldots\chi x_n), x_1^*\chi\ldots\chi x_d^*\otimes x_{d+1}^*\chi\ldots\chi x_{d+e}^*\otimes\cdots\otimes x_{d+e+\cdots+1}^*\chi\ldots\chi x_n^*\rangle=$$

$$\langle x_1\chi\ldots\chi x_d\otimes x_{d+1}\chi\ldots\chi x_{d+e}\otimes\cdots\otimes x_{d+e+\cdots+1}\chi\ldots\chi x_n, c_k(E^*)(x_1^*\chi\ldots\chi x_n^*)\rangle.$$

**Proof:** Using (14), and Proposition 19, we have

$$\langle x_1\chi\ldots\chi x_n, x_1^*\chi\ldots\chi x_n^*\rangle=$$

$$\sum_{\zeta\in M(\chi;n;d,e,\ldots,h)}\chi(\zeta)\langle x_{\zeta(1)}\chi\ldots\chi x_{\zeta(d)}, x_1^*\chi\ldots\chi x_d^*\rangle$$

$$\langle x_{\zeta(d+1)}\chi\ldots\chi x_{\zeta(d+e)}, x_{d+1}^*\chi\ldots\chi x_{d+e}^*\rangle$$

$$\cdots\langle x_{\zeta(d+e+\cdots+1)}\chi\ldots\chi x_{\zeta(n)}, x_{d+e+\cdots+1}^*\chi\ldots\chi x_n^*\rangle=$$

$$\sum_{\zeta\in M(\chi;n;d,e,\ldots,h)}\chi(\zeta)$$

$$\langle x_{\zeta(1)}\chi\ldots\chi x_{\zeta(d)}\otimes x_{\zeta(d+1)}\chi\ldots\chi x_{\zeta(d+e)}\otimes\cdots\otimes x_{\zeta(d+e+\cdots+1)}\chi\ldots\chi x_{\zeta(n)},$$

$$x_1^*\chi\ldots\chi x_d^*\otimes x_{d+1}^*\chi\ldots\chi x_{d+e}^*\otimes\cdots\otimes x_{d+e+\cdots+1}^*\chi\ldots\chi x_n^*\rangle=$$

$$\langle c_k(E)(x_1\chi\ldots\chi x_n), x_1^*\chi\ldots\chi x_d^*\otimes x_{d+1}^*\chi\ldots\chi x_{d+e}^*\otimes\cdots\otimes x_{d+e+\cdots+1}^*\chi\ldots\chi x_n^*\rangle.$$

Similarly, using the second identity of Proposition 19, we obtain the second identity of this corollary.

## 4 Coalgebra properties

Let us set $c_k = c_k(E)$, and $c_E = c_2(E)$, where $c_k(E)$, $k \geq 2$, is the homomorphism of graded $K$-modules from (15).

**Proposition 21** *One has*

$$c_k = (c_{k-1} \otimes 1) \circ c_E = (1 \otimes c_{k-1}) \circ c_E,$$

*where* 1 *is the identity map of* $[\chi](E)$.

**Proof:** We have

$$c_E(x_1\chi \ldots \chi x_n) = \sum_{p+h=n} \sum_{\rho \in M(\chi;n;p,h)} \chi(\rho)$$

$$(x_{\rho(1)}\chi \ldots \chi x_{\rho(p)}) \otimes (x_{\rho(p+1)}\chi \ldots \chi x_{\rho(n)}).$$

First, we apply the $K$-linear map $c_{k-1} \otimes 1$ and get

$$(c_{k-1} \otimes 1)(c_E(x_1\chi \ldots \chi x_n)) = \sum_{p+h=n} \sum_{\rho \in M(\chi;n;p,h)} \chi(\rho)$$

$$c_{k-1}(x_{\rho(1)}\chi \ldots \chi x_{\rho(p)}) \otimes (x_{\rho(p+1)}\chi \ldots \chi x_{\rho(n)}) =$$

$$\sum_{p+h=n} \sum_{\rho \in M(\chi;n;p,h)} \sum_{d+e+\cdots=p} \sum_{\varrho \in M(\chi;p;d,e,\ldots)} \chi(\rho\varrho)(x_{\rho(\varrho(1))}\chi \ldots \chi x_{\rho(\varrho(d))})$$

$$\otimes(x_{\rho(\varrho(d+1))}\chi \ldots \chi x_{\rho(\varrho(d+e))}) \otimes \ldots \otimes (x_{\rho(p+1)}\chi \ldots \chi x_{\rho(n)}) =$$

$$\sum_{p+h=n} \sum_{\rho \in M(\chi;n;p,h)} \sum_{d+e+\cdots=p} \sum_{\varrho \in M(\chi;p;d,e,\ldots)} \chi(\rho\varrho)(x_{\rho(\varrho(1))}\chi \ldots \chi x_{\rho(\varrho(d))})$$

$$\otimes(x_{\rho(\varrho(d+1))}\chi \ldots \chi x_{\rho(\varrho(d+e))}) \otimes \ldots \otimes (x_{\rho(\varrho(p+1))}\chi \ldots \chi x_{\rho(\varrho(n))}) =$$

$$\sum_{d+e+\cdots+h=n} \sum_{(\rho,\varrho) \in M(\chi;n;p,h) \times M(\chi;p;d,e,\ldots)} \chi(\rho\varrho)(x_{\rho(\varrho(1))}\chi \ldots \chi x_{\rho(\varrho(d))})$$

$$\otimes(x_{\rho(\varrho(d+1))}\chi \ldots \chi x_{\rho(\varrho(d+e))}) \otimes \ldots \otimes (x_{\rho(\varrho(p+1))}\chi \ldots \chi x_{\rho(\varrho(n))}).$$

In terms of Notation 29, we set $\rho\varrho\sigma'\tau' \ldots \eta' = 1 \cdot (\rho\varrho)$, where $\sigma' \in W_d$, $\tau' \in \omega^d(W_e)$,..., $\eta' \in \omega^p(W_h)$, $\sigma' = \sigma$, $\sigma \in W_d$, $\tau' = \omega^d(\tau)$, $\tau \in W_e$,..., $\eta' = \omega^p(\eta)$, $\eta \in W_h$. Then $\chi(\rho\varrho)\chi(\sigma)\chi(\tau) \ldots \chi(\eta) = \chi(1 \cdot (\rho\varrho))$, and we have

$$(c_{k-1} \otimes 1)(c_E(x_1\chi \ldots \chi x_n)) =$$

$$\sum_{d+e+\cdots+h=n} \sum_{(\rho,\varrho) \in M(\chi;n;p,h) \times M(\chi;p;d,e,\ldots)} \chi(1 \cdot (\rho\varrho))(x_{\rho(\varrho(\sigma(1)))}\chi \ldots \chi x_{\rho(\varrho(\sigma(d)))})$$

$$\otimes(x_{\rho(\varrho(\tau(d+1)))}\chi \ldots \chi x_{\rho(\varrho(\tau(d+e)))}) \otimes \ldots \otimes (x_{\rho(\varrho(\eta(p+1)))}\chi \ldots \chi x_{\rho(\varrho(n))}) =$$

$$\sum_{d+e+\cdots+h=n} \sum_{(\rho,\varrho) \in M(\chi;n;p,h) \times M(\chi;p;d,e,\ldots)} \chi(1 \cdot (\rho\varrho))(x_{(1 \cdot (\rho\varrho))(1)}\chi \ldots \chi x_{(1 \cdot (\rho\varrho))(d)})$$

$$\otimes(x_{(1\cdot(\rho\varrho))(d+1)}\chi\ldots\chi x_{(1\cdot(\rho\varrho))(d+e)})\otimes\ldots\otimes(x_{(1\cdot(\rho\varrho))(p+1)}\chi\ldots\chi x_{(1\cdot(\rho\varrho))(n)}).$$

According to Lemma 32 we obtain

$$(c_{k-1}\otimes 1)(c_E(x_1\chi\ldots\chi x_n)) =$$

$$\sum_{d+e+\cdots+h=n}\ \sum_{\varsigma\in M(\chi;n;d,e,\ldots,h)} \chi(\varsigma)(x_{\varsigma(1)}\chi\ldots\chi x_{\varsigma(d)})$$

$$\otimes(x_{\varsigma(d+1)}\chi\ldots\chi x_{\varsigma(d+e)})\otimes\ldots\otimes(x_{\varsigma(p+1)}\chi\ldots\chi x_{\varsigma(n)}) =$$

$$c_k(x_1\chi\ldots\chi x_n).$$

Similarly, we apply the $K$-linear map $1\otimes c_{k-1}$ and obtain

$$(1\otimes c_{k-1})(c_E(x_1\chi\ldots\chi x_n)) = \sum_{d+q=n}\ \sum_{\rho\in M(\chi;n;d,q)} \chi(\rho)$$

$$(x_{\rho(1)}\chi\ldots\chi x_{\rho(d)})\otimes c_{k-1}(x_{\rho(d+1)}\chi\ldots\chi x_{\rho(n)}) =$$

$$\sum_{d+q=n}\ \sum_{\rho\in M(\chi;n;d,q)}\ \sum_{e+\cdots+h=q}\ \sum_{\varrho\in\omega^d(M(\chi;q;e,\ldots,h))} \chi(\rho\varrho)(x_{\rho(1)}\chi\ldots\chi x_{\rho(d)})$$

$$\otimes(x_{\rho(\varrho(d+1))}\chi\ldots\chi x_{\rho(\varrho(d+e))})\otimes\ldots\otimes(x_{\rho(\varrho(p+1))}\chi\ldots\chi x_{\rho(\varrho(n))}) =$$

$$\sum_{d+q=n}\ \sum_{\rho\in M(\chi;n;d,q)}\ \sum_{e+\cdots+h=q}\ \sum_{\varrho\in\omega^d(M(\chi;q;e,\ldots,h))} \chi(\rho\varrho)(x_{\rho(\varrho(1))}\chi\ldots\chi x_{\rho(\varrho(d))})$$

$$\otimes(x_{\rho(\varrho(d+1))}\chi\ldots\chi x_{\rho(\varrho(d+e))})\otimes\ldots\otimes(x_{\rho(\varrho(p+1))}\chi\ldots\chi x_{\rho(\varrho(n))}) =$$

$$\sum_{d+e+\cdots+h=n}\ \sum_{(\rho,\varrho)\in M(\chi;n;d,q)\times\omega^d(M(\chi;q;e,\ldots,h))} \chi(\rho\varrho)(x_{\rho(\varrho(1))}\chi\ldots\chi x_{\rho(\varrho(d))})$$

$$\otimes(x_{\rho(\varrho(d+1))}\chi\ldots\chi x_{\rho(\varrho(d+e))})\otimes\ldots\otimes(x_{\rho(\varrho(p+1))}\chi\ldots\chi x_{\rho(\varrho(n))}) =$$

$$\sum_{d+e+\cdots+h=n}\ \sum_{(\rho,\varrho)\in M(\chi;n;d,q)\times\omega^d(M(\chi;q;e,\ldots,h))} \chi(1\cdot(\rho\varrho))(x_{(1\cdot(\rho\varrho))(1)}\chi\ldots\chi x_{(1\cdot(\rho\varrho))(d)})$$

$$\otimes(x_{(1\cdot(\rho\varrho))(d+1)}\chi\ldots\chi x_{(1\cdot(\rho\varrho))(d+e)})\otimes\ldots\otimes(x_{(1\cdot(\rho\varrho))(p+1)}\chi\ldots\chi x_{(1\cdot(\rho\varrho))(n)}) =$$

$$\sum_{d+e+\cdots+h=n}\ \sum_{\varsigma\in M(\chi;n;d,e,\ldots,h)} \chi(\varsigma)(x_{\varsigma(1)}\chi\ldots\chi x_{\varsigma(d)})$$

$$\otimes(x_{\varsigma(d+1)}\chi\ldots\chi x_{\varsigma(d+e)})\otimes\ldots\otimes(x_{\varsigma(p+1)}\chi\ldots\chi x_{\varsigma(n)}) =$$

$$c_k(x_1\chi\ldots\chi x_n).$$

Let us denote by $m_E$ the multiplication of the algebra $[\chi](E)$:

$$m_E\colon [\chi](E)\otimes[\chi](E) \to [\chi](E),$$

$$x_1\chi\ldots\chi x_d\otimes y_1\chi\ldots\chi y_e \mapsto x_1\chi\ldots\chi x_d\chi y_1\chi\ldots\chi y_e,$$

and by $\varepsilon_E\colon K \to [\chi](E)$, $\varepsilon_E(a) = a1$, the unit of the algebra $[\chi](E)$.

**Corollary 22** *(i) The $K$-linear map $c_E\colon[\chi](E)\to[\chi](E)\otimes[\chi](E)$ defines a structure of graded coassociative $K$-coalgebra on the graded $K$-module $[\chi](E)$, which is, moreover, counital, with counit, the linear form $\epsilon_E$ defined by the rule*

$$\epsilon_E\colon[\chi](E)\to K,$$

$$\epsilon_E(z)=\begin{cases} z & \text{if } z\in[\chi]^0(E)\\ 0 & \text{if } z\in([\chi](E))_+;\end{cases}$$

*(ii) The structure $([\chi](E),c_E,\epsilon_E)$ of graded coassociative $K$-coalgebra with counit on the graded $K$-module $[\chi](E)$ defines by functoriality a structure of graded associative algebra with unit on its dual $([\chi](E))^{*gr}=[\chi](E^*)$, and the last one coincide with the canonical structure $([\chi](E^*),m_{E^*},\varepsilon_{E^*})$ of graded associative algebra with unit on the graded $K$-module $[\chi](E^*)$;*

**Proof:** (i) The case $k=3$ of Proposition 21 yields coassociativity of $[\chi](E)$. We have

$$(\epsilon_E\otimes 1)(c_E(x_1\chi\ldots\chi x_n))=$$

$$(\epsilon_E\otimes 1)(\sum_{p+h=n}\sum_{\rho\in M(\chi;n;p,h)}\chi(\rho)(x_{\rho(1)}\chi\ldots\chi x_{\rho(p)})\otimes(x_{\rho(p+1)}\chi\ldots\chi x_{\rho(n)}))=$$

$$\sum_{p+h=n}\sum_{\rho\in M(\chi;n;p,h)}\chi(\rho)\epsilon_E((x_{\rho(1)}\chi\ldots\chi x_{\rho(p)}))\otimes(x_{\rho(p+1)}\chi\ldots\chi x_{\rho(n)})=$$

$$\sum_{\rho\in M(\chi;n;0,n)}\chi(\rho)\epsilon_E(1)\otimes(x_{\rho(1)}\chi\ldots\chi x_{\rho(n)})=$$

$$1\otimes x_1\chi\ldots\chi x_n=x_1\chi\ldots\chi x_n.$$

Similarly,

$$(1\otimes\epsilon_E)(c_E(x_1\chi\ldots\chi x_n))=$$

$$(1\otimes\epsilon_E)(\sum_{d+q=n}\sum_{\rho\in M(\chi;n;d,q)}\chi(\rho)(x_{\rho(1)}\chi\ldots\chi x_{\rho(d)})\otimes(x_{\rho(d+1)}\chi\ldots\chi x_{\rho(n)}))=$$

$$\sum_{d+q=n}\sum_{\rho\in M(\chi;n;d,q)}\chi(\rho)(x_{\rho(1)}\chi\ldots\chi x_{\rho(d)})\otimes\epsilon_E((x_{\rho(p+1)}\chi\ldots\chi x_{\rho(n)}))=$$

$$\sum_{\rho\in M(\chi;n;n,0)}\chi(\rho)(x_{\rho(1)}\chi\ldots\chi x_{\rho(n)})\otimes\epsilon_E(1)=$$

$$x_1\chi\ldots\chi x_n\otimes 1=x_1\chi\ldots\chi x_n.$$

Therefore

$$(\epsilon_E\otimes 1)\circ c_E=(1\otimes\epsilon_E)\circ c_E=1.$$

(ii) Corollary 20 yields that the multiplication $m_{E^*}$ in the graded algebra $([\chi](E^*),m_{E^*},\varepsilon_{E^*})$ is the transpose of the comultiplication $c_E$ of the graded coassociative $K$-coalgebra with counit $([\chi](E),c_E,\epsilon_E)$. Moreover, the counit $\epsilon_E$ is an element of $([\chi](E))^{*gr}$, such that if $z\in[\chi](E)$, $z=z_0+z_1+z_2+\cdots$, then $\langle z,\epsilon_E\rangle=z_0=z_0 1$. The transpose of $\epsilon_E$ is the $K$-linear map $K^*\to([\chi](E))^{*gr}$,

$\ell \mapsto \ell \circ \epsilon_E$. We compose it with the canonical isomorphism $K \to K^*$, and, after the identification of $([\chi](E))^{*gr}$ with $[\chi](E^*)$ via the isomorphism from Theorem 16, (i), we get the $K$-linear map $K \to [\chi](E^*)$, $k \mapsto k1$, and this is the unit 1 of the algebra $[\chi](E^*)$.

# 5 Inner products of a $\chi$-vector and a $\chi$-form

The semi-symmetric algebra $[\chi](E)$ becomes a $\mathbb{Z}$-graded $K$-module by setting $[\chi]^d(E) = 0$ for negative integers $d$.

Let $d$ and $q \geq 0$ be integers with $d + q = n$. Let $a = a_1\chi \dots \chi a_q$ be a fixed decomposable $q - \chi$-vector. The right multiplication by $a$ in the algebra $[\chi](E)$,

$$x_1\chi \dots \chi x_d \mapsto x_1\chi \dots \chi x_d \chi a_1 \chi \dots \chi a_q,$$

defines an endomorphism $e'(a)$ of degree $q$ of the $\mathbb{Z}$-graded $K$-module $[\chi](E)$. The transpose of $e'(a)$ is an endomorphism $i'(a)$ of degree $-q$ of the dual $\mathbb{Z}$-graded $K$-module $[\chi](E^*)$. We define $e'(a)$ and $i'(a)$ for $a \in [\chi](E)$ by linearity.

For any $\chi$-vector $a \in [\chi](E)$ and for any $\chi$-form $a^* \in [\chi](E^*)$ denote the $\chi$-form $i'(a)(a^*)$ by $a\rfloor a^*$ and call it *left inner product of $a$ and $a^*$*. Thus,

$$\langle x\chi a, a^*\rangle = \langle x, a\rfloor a^*\rangle$$

for $x \in [\chi](E)$.

**Proposition 23** *Let $d$ and $q \geq 0$ be integers with non-negative sum $n = d + q$. Then for any decomposable $q - \chi$-vector $a = a_1\chi \dots \chi a_q$, and for any decomposable $n - \chi$-form $a^* = a_1^*\chi \dots \chi a_n^*$, the left inner product $a\rfloor a^*$ is the $d - \chi$-linear form*

$$\sum_{\rho \in M(\chi;n;d,q)} \chi(\rho)\langle a_1\chi \dots \chi a_q, a^*_{\rho(d+1)}\chi \dots \chi a^*_{\rho(n)}\rangle a^*_{\rho(1)}\chi \dots \chi a^*_{\rho(d)}$$

*in case $n \geq q$, and $0$ in case $n < q$.*

**Proof:** In case $n < q$ we have $a\rfloor a^* = 0$ by the definition of the endomorphism $i'(a)$. Otherwise, $i'(a_1\chi \dots \chi a_q)(a_1^*\chi \dots \chi a_n^*)$ is the linear form

$$x_1\chi \dots \chi x_d \mapsto \langle x_1\chi \dots \chi x_d\chi a_1\chi \dots \chi a_q, a_1^*\chi \dots \chi a_n^*\rangle$$

on $[\chi](E)$. Proposition 19 yields

$$\langle x_1\chi \dots \chi x_d\chi a_1\chi \dots \chi a_q, a_1^*\chi \dots \chi a_n^*\rangle =$$

$$\sum_{\rho \in M(\chi;n;d,q)} \chi(\rho)\langle x_1\chi \dots \chi x_d, a^*_{\rho(1)}\chi \dots \chi a^*_{\rho(d)}\rangle\langle a_1\chi \dots \chi a_q, a^*_{\rho(d+1)}\chi \dots \chi a^*_{\rho(n)}\rangle =$$

$$\langle x_1\chi \dots \chi x_d, \sum_{\rho \in M(\chi;n;d,q)} \chi(\rho)\langle a_1\chi \dots \chi a_q, a^*_{\rho(d+1)}\chi \dots \chi a^*_{\rho(n)}\rangle a^*_{\rho(1)}\chi \dots \chi a^*_{\rho(d)}\rangle.$$

After the identification of $[\chi]^d(E))^{*gr}$ with $[\chi](E^*)$, we obtain the result.

Given non-negative integers $d$, $q$ with $d+q=n$, a integer $m \geq 1$, and $i \in J(\chi, m, d)$, $j \in J(\chi, m, q)$, $k \in J(\chi, m, n)$, one sets

$$M_{k,.,j}(\chi; n; d, q) = \{\rho \in M(\chi; n; d, q) \mid j_1 = k_{\rho(d+1)}, \ldots, j_q = k_{\rho(n)}\},$$

$$M'_{k,i,.}(\chi; n; d, q) = \{\rho \in M(\chi; n; d, q) \mid k_{\rho(1)} = i_1, \ldots, k_{\rho(d)} = i_d\}.$$

**Corollary 24** *Let $(e_\ell)_{\ell=1}^m$ be a basis for the $K$-module $E$ and let $(e_\ell^*)_{\ell=1}^m$ be its dual basis in the dual $K$-module $E^*$. Let $(e_j)_{j \in J(\chi,m)}$ and $(e_k^*)_{k \in J(\chi,m)}$ be the corresponding bases of $[\chi](E)$ and $[\chi](E^*)$, respectively. If $j \in J(\chi, m, q)$, $k \in J(\chi, m, n)$, and if $d+q=n$, then the left inner product $e_j \rfloor e_k^*$ is the $d-\chi$-linear form*

$$\sum_{\rho \in M_{k,.,j}(\chi;n;d,q)} \chi(\rho) e^*_{\rho(1)} \chi \ldots \chi e^*_{\rho(d)}$$

*in case $n \geq q$, and $0$ in case $n < q$.*

**Proof:** In accord with Proposition 23, in case $n < q$ we have $e_j \rfloor e_k^* = 0$, and in case $n \geq q$, we have

$$e_j \rfloor e_k^* =$$

$$\sum_{\rho \in M(\chi;n;d,q)} \chi(\rho) \langle e_{j_1} \chi \ldots \chi e_{j_q}, e^*_{k_{\rho(d+1)}} \chi \ldots \chi e^*_{k_{\rho(n)}} \rangle e^*_{k_{\rho(1)}} \chi \ldots \chi e^*_{k_{\rho(d)}} =$$

$$\sum_{\rho \in M_{k,.,j}(\chi;n;d,q)} \chi(\rho) e^*_{k_{\rho(1)}} \chi \ldots \chi e^*_{k_{\rho(d)}}.$$

**Proposition 25** *The addition and the external composition law $(a, a^*) \mapsto a \rfloor a^*$ on $[\chi](E^*)$ define on this set a structure of left unital $[\chi](E)$-module.*

**Proof:** The external composition law is bilinear and the associativity of the the graded algebra $[\chi](E)$ is equivalent to the equality $e'(a\chi b) = e'(b) \circ e'(a)$ for $a, b \in [\chi](E)$. Then $i'(a\chi b) = i'(a) \circ i'(b)$, and hence $(a\chi b) \rfloor a^* = a \rfloor (b \rfloor a^*)$. Moreover, $1 \rfloor a^* = a^*$.

Let $p \geq 0$ and $h$ be integers with $p+h=n$. Let $a^* = a_1^* \chi \ldots \chi a_p^*$ be a fixed decomposable $p-\chi$-form. The left multiplication by $a^*$ in the algebra $[\chi](E^*)$,

$$x_1^* \chi \ldots \chi x_h^* \mapsto a_1^* \chi \ldots \chi a_p^* \chi x_1^* \chi \ldots \chi x_h^*,$$

defines an endomorphism $e(a^*)$ of degree $p$ of the $\mathbb{Z}$-graded $K$-module $[\chi](E^*)$. The transpose of $e(a^*)$ is an endomorphism $i(a^*)$ of degree $-p$ of the $\mathbb{Z}$-graded $K$-module $[\chi](E)$. We define $e(a^*)$ and $i(a^*)$ for $a^* \in [\chi](E)$ by linearity.

For any $\chi$-form $a^* \in [\chi](E^*)$, and for any $\chi$-vector $a \in [\chi](E)$ denote the $\chi$-vector $i(a^*)(a)$ by $a \lfloor a^*$ and call it *right inner product of $a$ and $a^*$*. Thus,

$$\langle a \lfloor a^*, x^* \rangle = \langle a, a^* \chi x^* \rangle$$

for $x^* \in [\chi](E^*)$.

**Proposition 26** *Let $h$ and $p \geq 0$ be integers with non-negative sum $n = p + h$. Then for any decomposable $n-\chi$-vector $a = a_1\chi \dots \chi a_n$, and for any decomposable $p-\chi$-form $a^* = a_1^*\chi \dots \chi a_p^*$, the right inner product $a\lfloor a^*$ is the $h-\chi$-vector*

$$\sum_{\rho \in M(\chi;n;p,h)} \chi(\rho)\langle a_{\rho(1)}\chi \dots \chi a_{\rho(p)}, a_1^*\chi \dots \chi a_p^*\rangle a_{\rho(p+1)}\chi \dots \chi a_{\rho(n)}$$

*in case $n \geq p$, and $0$ in case $n < p$.*

**Proof:** In case $n < p$ we have $a\lfloor a^* = 0$ by the definition of the endomorphism $i(a^*)$. Otherwise, according to Proposition 19 we have

$$\langle a\lfloor a^*, x_1^*\chi \dots \chi x_h^*\rangle = \langle a_1\chi \dots \chi a_n, a_1^*\chi \dots \chi a_p^*\chi x_1^*\chi \dots \chi x_h^*\rangle =$$

$$\sum_{\rho \in M(\chi;n;p,h)} \chi(\rho)\langle a_{\rho(1)}\chi \dots \chi a_{\rho(p)}, a_1^*\chi \dots \chi a_p^*\rangle\langle a_{\rho(p+1)}\chi \dots \chi a_{\rho(n)}, x_1^*\chi \dots \chi x_h^*\rangle =$$

$$\langle \sum_{\rho \in M(\chi;n;p,h)} \chi(\rho)\langle a_{\rho(1)}\chi \dots \chi a_{\rho(p)}, a_1^*\chi \dots \chi a_p^*\rangle a_{\rho(p+1)}\chi \dots \chi a_{\rho(n)}, x_1^*\chi \dots \chi x_h^*\rangle,$$

and we get the result.

**Corollary 27** *Let $(e_\ell)_{\ell=1}^m$ be a basis for the $K$-module $E$ and let $(e_\ell^*)_{\ell=1}^m$ be its dual basis in the dual $K$-module $E^*$. Let $(e_j)_{j\in J(\chi,m)}$ and $(e_k^*)_{k\in J(\chi,m)}$ be the corresponding bases of $[\chi](E)$ and $[\chi](E^*)$, respectively. If $j \in J(\chi, m, p)$, $k \in J(\chi, m, n)$, and if $p + h = n$, then the right inner product $e_k\lfloor e_j^*$ is the $h - \chi$-vector*

$$\sum_{\rho \in M_{k,j,.}(\chi;n;p,h)} \chi(\rho)e_{k_{\rho(p+1)}}\chi \dots \chi e_{k_{\rho(n)}}$$

*in case $n \geq p$, and $0$ in case $n < p$.*

**Proof:** In accord with Proposition 23, in case $n < p$ we have $e_k\lfloor e_j^* = 0$, and in case $n \geq p$, we have

$$e_k\lfloor e_j^* =$$

$$\sum_{\rho \in M(\chi;n;p,h)} \chi(\rho)\langle e_{k_{\rho(1)}}\chi \dots \chi e_{k_{\rho(p)}}, e_{j_1}^*\chi \dots \chi e_{j_p}^*\rangle e_{k_{\rho(p+1)}}\chi \dots \chi e_{k_{\rho(n)}} =$$

$$\sum_{\rho \in M_{k,j,.}(\chi;n;p,h)} \chi(\rho)e_{k_{\rho(p+1)}}\chi \dots \chi e_{k_{\rho(n)}}.$$

**Proposition 28** *The addition and the external composition law $(a, a^*) \mapsto a\lfloor a^*$ on $[\chi](E)$ define on this set a structure of right unital $[\chi](E^*)$-module.*

**Proof:** The external composition law is bilinear and the associativity of the the graded algebra $[\chi](E^*)$ is equivalent to the equality $e(a^*\chi b^*) = e(a^*) \circ e(b^*)$ for $a^*, b^* \in [\chi](E^*)$. Then $i(a^*\chi b^*) = i(b^*)\circ i(a^*)$, and hence $a\lfloor(a^*\chi b^*) = (a\lfloor a^*)\lfloor b^*$. Moreover, $a\lfloor 1 = a$.

## 6 Acknowledgements

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## A Appendix

**Notation 29** *Let $d$, $e$,..., $h$ be $k$ in number nonnegative integers with $d + e + \cdots + h = n$. We assume $k \leq n$. Let $\alpha\colon [1,d] \to [1,n]$, $\beta\colon [1,e] \to [1,n]$,..., $\gamma\colon [1,h] \to [1,n]$, be strictly increasing maps with disjoint images. Let $\theta_\alpha \in S_n$ be a permutation with $\theta_\alpha(1) = \alpha(1)$,..., $\theta_\alpha(d) = \alpha(d)$, let $\theta_\beta \in S_n$ be a permutation with $\theta_\beta(1) = \beta(1)$,..., $\theta_\beta(e) = \beta(e)$,..., let $\theta_\gamma \in S_n$ be a permutation with $\theta_\gamma(1) = \gamma(1)$,..., $\theta_\gamma(h) = \gamma(h)$. For any permutation $\theta \in S_n$ we denote by $c_\theta\colon S_n \to S_n$ the conjugation $c_\theta(\zeta) = \theta\zeta\theta^{-1}$. We have*

$$c_{\theta_\alpha}(S_d) = S_{Im\alpha},\ c_{\theta_\beta}(S_e) = S_{Im\beta},\ \ \ldots, c_{\theta_\gamma}(S_h) = S_{Im\gamma}.$$

*Let $K$ be a commutative ring with unit 1. Let $U \leq S_d$, $V \leq S_e$,... ,$W \leq S_h$ be permutation groups, and let $\varepsilon\colon U \to U(K)$, $\delta\colon V \to U(K)$,..., $\varpi\colon W \to U(K)$, be linear $K$-valued characters. We embed the Cartesian product $U \times V \times \cdots \times W$ in $S_n$ as $X = c_{\theta_\alpha}(U)c_{\theta_\beta}(V)\ldots c_{\theta_\gamma}(W)$ and for any $\zeta \in X$, $\zeta = c_{\theta_\alpha}(\sigma)c_{\theta_\beta}(\tau)\ldots c_{\theta_\gamma}(\eta)$, $\sigma \in U$, $\tau \in V$,..., $\eta \in W$, we set*

$$\chi(\zeta) = \varepsilon(\sigma)\delta(\tau)\ldots\varpi(\eta).$$

*The map $\chi\colon X \to U(K)$ is a $K$-linear character of the group $X$. Let $E$ be a $K$-module and let $(x_1, \ldots, x_d) \in E^d$, $(y_1, \ldots, y_e) \in E^e$,..., $(z_1, \ldots, z_h) \in E^h$ be generic elements. We set*

$$\xi_i = \begin{cases} x_{\alpha^{-1}(i)} & \text{if } i \in Im\alpha \\ y_{\beta^{-1}(i)} & \text{if } i \in Im\beta \\ \vdots & \vdots \\ z_{\gamma^{-1}(i)} & \text{if } i \in Im\gamma \end{cases}$$

*Let $Y \leq S_n$ be a permutation group with $X \leq Y$, and let $M^{\alpha,\beta,\ldots,\gamma}_{U,V,\ldots,W}(Y)$ be the set of all lexicographically minimal representatives of the left classes of $Y$ modulo $X$. For any $\zeta' \in Y$, $\zeta \in M^{\alpha,\beta,\ldots,\gamma}_{U,V,\ldots,W}(Y)$, we denote by $\zeta' \cdot \zeta$ the lexicographically minimal representative of $\zeta'\zeta$ modulo $X$, and set $\zeta' \cdot \zeta = \zeta'\zeta\upsilon_{\zeta'\zeta}$, where $\upsilon_{\zeta'\zeta} \in X$, $\upsilon_{\zeta'\zeta} = c_{\theta_\alpha}(\sigma)c_{\theta_\beta}(\tau)\ldots c_{\theta_\gamma}(\eta)$, with $\sigma \in U$, $\tau \in V$,..., $\eta \in W$.*

*In case an $\omega$-invariant sequence of characters $\chi = (\chi_d)_{d\geq 1}$ is given, if the opposite is not stated, we specialize the maps $\alpha, \beta, \ldots, \gamma$, the groups $U, V, \ldots, W$, and the characters $\varepsilon, \delta, \ldots, \varpi$, on them, as follows: $\alpha(1) = 1$,..., $\alpha(d) = d$, $\beta(1) = d+1$,..., $\beta(e) = d+e$,..., $\gamma(1) = d+e+\cdots+1$,..., $\gamma(h) = d+e+\cdots+h$, $U = W_d$, $V = W_e$, ..., $W = W_h$, $Y = W_n$, $\varepsilon = \chi_d$, $\delta = \chi_e$,..., $\varpi = \chi_h$. Then*

$$c_{\theta_\alpha}(U) = W_d,\ c_{\theta_\beta}(V) = \omega^d(W_e),\ \ \ldots, c_{\theta_\gamma}(W) = \omega^{d+e+\cdots}(W_h),$$

*and, using notation from Remark 17,*

$$M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)=M(\chi;n;d,e,\dots,h).$$

**Lemma 30** *The rule $(\zeta',\zeta)\mapsto\zeta'\cdot\zeta$ defines a left action of the group $Y$ on the set $M(Y;\alpha,\beta,\dots,\gamma)$.*

**Proof:** Let $\zeta''\in Y$. The three elements $(\zeta''\zeta')\cdot\zeta$, $\zeta''(\zeta'\cdot\zeta)$, and $\zeta''\cdot(\zeta'\cdot\zeta)$ are in the class $\zeta''\zeta'\zeta X$, so we get $(\zeta''\zeta')\cdot\zeta=\zeta''\cdot(\zeta'\cdot\zeta)$. Finally, $1_Y\cdot\zeta=\zeta$.

**Lemma 31** *Let $\pi$ be a linear $K$-valued character of $Y$, and $\pi_{|X}=\chi$. Let $\varepsilon^2=1_U$, $\delta^2=1_V$,..., $\varpi^2=1_W$, and $\pi^2=1_Y$. The formula*

$$[\pi]^n(E)\to\coprod_{d+e+\dots+h=n}[\varepsilon]^d(E)\otimes[\delta]^e(E)\dots\otimes[\varpi]^h(E),$$

$$\xi_1\pi\dots\pi\xi_n\mapsto\sum_{d+e+\dots+h=n}\ \sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta)$$

$$(\xi_{\zeta(\alpha(1))}\varepsilon\dots\varepsilon\xi_{\zeta(\alpha(d))})\otimes(\xi_{\zeta(\beta(1))}\delta\dots\delta\xi_{\zeta(\beta(e))})\otimes\dots\otimes(\xi_{\zeta(\gamma(1))}\varpi\dots\varpi\xi_{\zeta(\gamma(h))}),$$

*defines a $K$-linear map.*

**Proof:** The map

$$f\colon E^n\to\coprod_{d+e+\dots+h=n}[\varepsilon]^d(E)\otimes[\delta]^e(E)\dots\otimes[\varpi]^h(E),$$

$$f(\xi_1,\dots,\xi_n)=\sum_{d+e+\dots+h=n}\ \sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta)$$

$$(\xi_{\zeta(\alpha(1))}\varepsilon\dots\varepsilon\xi_{\zeta(\alpha(d))})\otimes(\xi_{\zeta(\beta(1))}\delta\dots\delta\xi_{\zeta(\beta(e))})\otimes\dots\otimes(\xi_{\zeta(\gamma(1))}\varpi\dots\varpi\xi_{\zeta(\gamma(h))}),$$

is multilinear and semi-symmetric of weight $\pi$. Indeed, let $\zeta'\in Y$. We have

$$f(\xi_{\zeta'(1)},\dots,\xi_{\zeta'(n)})=\sum_{d+e+\dots+h=n}\ \sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta)$$

$$(\xi_{\zeta'(\zeta(\alpha(1)))}\varepsilon\dots\varepsilon\xi_{\zeta'(\zeta(\alpha(d)))})\otimes(\xi_{\zeta'(\zeta(\beta(1)))}\delta\dots\delta\xi_{\zeta'(\zeta(\beta(e)))})\otimes\dots$$
$$\otimes(\xi_{\zeta'(\zeta(\gamma(1)))}\varpi\dots\varpi\xi_{\zeta'(\zeta(\gamma(h)))})=$$
$$\pi(\zeta')\sum_{d+e+\dots+h=n}\ \sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta'\zeta)$$

$$(\xi_{\zeta'(\zeta(\alpha(1)))}\varepsilon\dots\varepsilon\xi_{\zeta'(\zeta(\alpha(d)))})\otimes(\xi_{\zeta'(\zeta(\beta(1)))}\delta\dots\delta\xi_{\zeta'(\zeta(\beta(e)))})\otimes\dots$$
$$\otimes(\xi_{\zeta'(\zeta(\gamma(1)))}\varpi\dots\varpi\xi_{\zeta'(\zeta(\gamma(h)))}).$$

Since

$$\pi(\zeta'\cdot\zeta)=\pi(\zeta'\zeta\upsilon_{\zeta'\zeta})=\pi(\zeta'\zeta)\chi(\upsilon_{\zeta'\zeta})=\pi(\zeta'\zeta)\varepsilon(\sigma)\delta(\tau)\dots\varpi(\eta),$$

using Lemma 30, we have

$$f(\xi_{\zeta'(1)},\dots,\xi_{\zeta'(n)})=\pi(\zeta')\sum_{d+e+\dots+h=n}\sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta'\zeta)$$

$$(\xi_{\zeta'(\zeta(\alpha(1)))}\varepsilon\dots\varepsilon\xi_{\zeta'(\zeta(\alpha(d)))})\otimes(\xi_{\zeta'(\zeta(\beta(1)))}\delta\dots\delta\xi_{\zeta'(\zeta(\beta(e)))})\otimes\dots$$
$$\otimes(\xi_{\zeta'(\zeta(\gamma(1)))}\varpi\dots\varpi\xi_{\zeta'(\zeta(\gamma(h)))})=$$
$$\pi(\zeta')\sum_{d+e+\dots+h=n}\sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta'\cdot\zeta)$$
$$\varepsilon(\sigma)(\xi_{\zeta'(\zeta(\alpha(1)))}\varepsilon\dots\varepsilon\xi_{\zeta'(\zeta(\alpha(d)))})\otimes\delta(\tau)(\xi_{\zeta'(\zeta(\beta(1)))}\delta\dots\delta\xi_{\zeta'(\zeta(\beta(e)))})\otimes\dots$$
$$\otimes\varpi(\eta)(\xi_{\zeta'(\zeta(\gamma(1)))}\varpi\dots\varpi\xi_{\zeta'(\zeta(\gamma(h)))})=$$
$$\pi(\zeta')\sum_{d+e+\dots+h=n}\sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta'\cdot\zeta)$$
$$(\xi_{\zeta'(\zeta(\alpha(\sigma(1))))}\varepsilon\dots\varepsilon\xi_{\zeta'(\zeta(\alpha(\sigma(d))))})\otimes(\xi_{\zeta'(\zeta(\beta(\tau(1))))}\delta\dots\delta\xi_{\zeta'(\zeta(\beta(\tau(e))))})\otimes\dots$$
$$\otimes\varpi(\eta)(\xi_{\zeta'(\zeta(\gamma(\eta(1))))}\varpi\dots\varpi\xi_{\zeta'(\zeta(\gamma(\eta(h))))})=$$
$$\pi(\zeta')\sum_{d+e+\dots+h=n}\sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta'\cdot\zeta)$$
$$(\xi_{\zeta'(\zeta(\upsilon_{\zeta'\zeta}(\alpha(1))))}\varepsilon\dots\varepsilon\xi_{\zeta'(\zeta(\upsilon_{\zeta'\zeta}(\alpha(d))))})\otimes(\xi_{\zeta'(\zeta(\upsilon_{\zeta'\zeta}(\beta(1))))}\delta\dots\delta\xi_{\zeta'(\zeta(\upsilon_{\zeta'\zeta}(\beta(e))))})\otimes\dots$$
$$\otimes(\xi_{\zeta'(\zeta(\upsilon_{\zeta'\zeta}(\gamma(1))))}\varpi\dots\varpi\xi_{\zeta'(\zeta(\upsilon_{\zeta'\zeta}(\gamma(h))))})=$$
$$\pi(\zeta')\sum_{d+e+\dots+h=n}\sum_{\zeta\in M^{\alpha,\beta,\dots,\gamma}_{U,V,\dots,W}(Y)}\pi(\zeta'\cdot\zeta)$$
$$(\xi_{(\zeta'\cdot\zeta)(\alpha(1))}\varepsilon\dots\varepsilon\xi_{(\zeta'\cdot\zeta)(\alpha(d))})\otimes(\xi_{(\zeta'\cdot\zeta)(\beta(1))}\delta\dots\delta\xi_{(\zeta'\cdot\zeta)(\beta(e))})\otimes\dots$$
$$\otimes(\xi_{(\zeta'\cdot\zeta)(\gamma(1))}\varpi\dots\varpi\xi_{(\zeta'\cdot\zeta)(\gamma(h))})=$$
$$\pi(\zeta')f(\xi_1,\dots,\xi_n).$$

Therefore, according to [3, (1.1.1)], $f$ gives rise to the desired $K$-linear map.

Let $\chi=(\chi_d)_{d\geq1}$ be an $\omega$-invariant sequence of characters. Using Notation 29, we have

**Lemma 32** *The maps*

$$M(\chi;n;p,h)\times M(\chi;p;d,e,\dots)\to M(\chi;n;d,e,\dots,h),$$
$$M(\chi;n;d,q)\times\omega^dM(\chi;q;e,\dots,h)\to M(\chi;n;d,e,\dots,h),$$
$$(\rho,\varrho)\mapsto 1\cdot(\rho\varrho),$$

*are bijections.*

**Proof:** If $W_n/W_p \times \omega^p(W_h)$ is a set of representatives of the left classes of $W_n$ modulo $W_p \times \omega^p(W_h)$, if $W_p \times \omega^p(W_h)/W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$ is a set of representatives of the left classes of $W_p \times \omega^p(W_h)$ modulo $W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$, then the family

$$\{\rho\varrho \mid (\rho, \varrho) \in (W_n/W_p \times \omega^p(W_h)) \times (W_p \times \omega^p(W_h)/W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h))\}$$

of elements of $W_n$ is a set of representatives of the left classes of $W_n$ modulo $W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$. Thus, the first map is a bijection because $M\,(\chi; p; d, e, \ldots)$ is a set of representatives of the left classes of $W_p \times \omega^p(W_h)$ modulo $W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$. Similarly, if $W_n/W_d \times \omega^d(W_q)$ is a set of representatives of the left classes of $W_n$ modulo $W_d \times \omega^d(W_q)$, if $W_d \times \omega^d(W_q)/W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$ is a set of representatives of the left classes of $W_d \times \omega^d(W_q)$ modulo $W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$, then the family

$$\{\rho\varrho \mid (\rho, \varrho) \in (W_n/W_d \times \omega^d(W_q)) \times (W_d \times \omega^d(W_q)/W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h))\}$$

of elements of $W_n$ is a set of representatives of the left classes of $W_n$ modulo $W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$. The second map is a bijection, too, because $\omega^d M\,(\chi; q; e, \ldots, h)$ is a set of representatives of the left classes of $W_d \times \omega^d(W_q)$ modulo $W_d \times \omega^d(W_e) \times \cdots \times \omega^p(W_h)$.

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